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### Notation

$\mathbb{Z}$  = the set of integers

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{R}$  = the set of real numbers

$\mathbb{Q}$  = the set of rational numbers

$\mathbb{C}$  = the set of complex numbers

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1. Let  $\mathcal{H}_1 \subseteq \mathcal{H}_2$  be complex Hilbert spaces and  $\phi : \mathcal{H}_1 \rightarrow \mathbb{C}$  be a bounded linear functional. Show that the Hahn-Banach norm preserving extension of  $\phi$  to  $\mathcal{H}_2$  is unique.

2. Does there exist a function  $f$  such that  $f^2 \in L^1[0, 1]$  but  $f \notin L^2[0, 1]$ ? Here the associated measure is the Lebesgue measure on  $[0, 1]$ .

3. For  $n \geq 1$ , define

$$f_n(x) = \frac{n^2}{(n^2 + x^2)},$$

for all  $x \in \mathbb{R}$ . Prove that  $\{f_n\}$  converges uniformly on bounded intervals but  $\{f_n\}$  does not converge uniformly on  $\mathbb{R}$ .

4. Let  $f$  be an entire function such that

$$f(z + 1) = f(z) + 1, \quad f(z + i) = f(z) + i,$$

for all  $z \in \mathbb{C}$ . Show that there is a constant  $c \in \mathbb{C}$  such that

$$f(z) = z + c,$$

for all  $z \in \mathbb{C}$ .

5. Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Prove that there is no continuous function  $f$  on  $\overline{\mathbb{D}}$  such that  $f$  is holomorphic on  $\mathbb{D}$  and

$$f(z) = \bar{z},$$

for all  $z \in \partial\mathbb{D}$ .

6. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous with

$$\lim_{x \rightarrow \infty} f(x) = a, \quad \lim_{x \rightarrow -\infty} f(x) = b,$$

with  $a, b$  finite. Prove that  $f$  is uniformly continuous.

7. Denote by  $m$  the Lebesgue measure on  $\mathbb{R}$ . Let  $S \subseteq \mathbb{R}$  be a Lebesgue measurable subset. If

$$m(S) = 2024,$$

then prove that there exists a Lebesgue measurable set  $T \subseteq S$  such that

$$m(T) = \frac{1}{2024}.$$

8. Consider the differential equation

$$y'' + Py' + Qy = 0,$$

where  $P$  and  $Q$  are continuous functions on  $[0, 1]$ . For a nonzero solution  $y$  of the above differential equation, define

$$Z(y) = \{x \in [0, 1] \mid y(x) = 0\}.$$

Prove that for any two nonzero solutions  $y_1$  and  $y_2$ , either

$$Z(y_1) = Z(y_2) \quad \text{or} \quad Z(y_1) \cap Z(y_2) = \emptyset.$$