- **The symbol $\mathbb{R}$ will denote the set of all real numbers.**
- **The symbol $\mathbb{C}$ will denote the set of all complex numbers.**
- **The symbol i will denote the square root of -1.**

1. Suppose $f : (0, \infty) \to \mathbb{R}$ is a twice differentiable function satisfying the following conditions for all $x \in (0, \infty)$:

$$f(x) \geq 0, \ f'(x) \leq 0, \ f''(x) \geq 0.$$

Show that $\lim_{x \to \infty} f'(x) = 0$.

2. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

   (a) For all $n \geq 1$, the n-th order derivative $f^{(n)}(x)$ exists for all $x \in \mathbb{R}$.

   (b) For all $x \neq 0$, $|f(x)| \leq e^{-\frac{1}{x^2}}$.

Prove that $f(0) = 0$ and $f^{(n)}(0) = 0$ for all $n \geq 1$.

3. For $y \in \mathbb{R}$ and $M > 0$, let $\gamma_{y,M} : [-M, M] \to \mathbb{C}$ be the curve

$$\gamma_{y,M}(x) = x + iy$$

for all $x \in [-M, M]$.

   (a) Show that for all $y \in \mathbb{R}$, the limit

$$I(y) := \lim_{M \to \infty} \int_{\gamma_{y,M}} e^{-z^2} dz$$

   exists.

   (b) Show that for all $y_1, y_2 \in \mathbb{R}$, $I(y_1) = I(y_2)$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function of the form

$$f(z) = u(z) + iv(z),$$

where u and v are real-valued functions defined on $\mathbb{C}$ which we identify with $\mathbb{R}^2$.

(a) Prove that for all $x, y \in \mathbb{R}$,

$$f'(x + iy) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

(b) Assume that for all $\alpha \in \mathbb{R}$, there exists a real number β (depending on α) such that

$$f(\{z : \mathrm{Re}(z) = \alpha\}) \subseteq \{z : \mathrm{Re}(z) = \beta\}.$$

Here, $\mathrm{Re}(z)$ denotes the real part of the complex number z.

Show that there are constants $a \in \mathbb{R}$ and $b \in \mathbb{C}$ such that for all $z \in \mathbb{C}$,

$$f(z) = az + b.$$

5. Let $\alpha > 2$ be a real number.

Prove that the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \int_0^\infty \frac{\cos(tx)}{1 + x^\alpha} dx$$

is differentiable.

6. Consider the Banach space $L^1(\mathbb{R})$ equipped with the norm

$$\|f\| := \int_{\mathbb{R}} |f(x)|\, dx,$$

where dx denotes the Lebesgue measure on $\mathbb{R}$.

Let $\phi \in L^1(\mathbb{R})$ be such that $\phi(x) \geq 0$ almost everywhere. For $f \in L^1(\mathbb{R})$, define a measurable function $T(f)$ on $\mathbb{R}$ as

$$T(f)(x) = \int_{\mathbb{R}} \phi(x-y)f(y)dy$$

almost everywhere.

(a) Prove that $T(f)$ belongs to $L^1(\mathbb{R})$ for all f in $L^1(\mathbb{R})$.

(b) Prove that the linear map T is a bounded linear operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

(c) Compute the operator norm of T.

7. Suppose X, Y are Banach spaces and $T : X \to Y$ is a bounded linear map such that the dimension of the vector space

$$\text{Im}(T) := \{Tx : x \in X\} \subseteq Y$$

is finite. Moreover, assume that $\{x_n\}_{n\geq 1}$ is a sequence in X such that for all bounded linear functionals f on X, $\lim_{n\to\infty} f(x_n) = 0$.

(a) Prove that for all bounded linear functionals g on Y, $\lim_{n\to\infty} g(T(x_n)) = 0$.

(b) Prove that $\lim_{n\to\infty} \|T(x_n)\| = 0$.

8. For a normed linear space N, the symbol N^* will stand for the set of all bounded linear functionals on N.

   Let E and F be Banach spaces and let $T : E \to F$ is a linear map such that there exists another linear map $S : F^* \to E^*$ satisfying the equation

$$g(T(e)) = S(g)(e)$$

   for all $e \in E$ and for all $g \in F^*$.

   Prove that T is a bounded linear map.