- $\mathbb{N}$ = natural numbers • $\mathbb{Z}$ = integers • $\mathbb{Q}$ = rational numbers
- $\mathbb{R}$ = real numbers • $\mathbb{C}$ = complex numbers

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1. Suppose $\{f_n : (0,1) \to \mathbb{R}\}_{n \geq 1}$ is a sequence of continuous functions such that for any Cauchy sequence $\{x_n\}$ in $(0,1)$, we have $f_n(x_n) \to 0$ as $n \to \infty$.

   (a) Find the pointwise limit of $\{f_n\}$.

   (b) Is the above convergence uniform? Justify your answer.

2. Find all the real values of p such that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right)^p$$

   is convergent.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Is f differentiable at 0 ? Justify your answer.

   (b) Show that there is no complex valued function $g : U \to \mathbb{C}$ defined on an open subset U of $\mathbb{C}$ containing the real line such that g is holomorphic on U and its restriction to $\mathbb{R}$ is f.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that for each $z \in \mathbb{C}$, there is an $n \in \mathbb{N}$ (this n may depend on z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial.

5. Compute, with justification, the following limit:

$$\lim_{n \to \infty} \int_{\log n}^{e^n} x^{2022} \, e^{-x^9} \, dx \,.$$

6. Let $f : [0, \infty) \to [0, \infty)$ be continuous and $\int_0^\infty f(x)dx < \infty$. Prove that

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n xf(x)dx = 0.$$

7. Show that the solutions to the system of differential equations

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = -x$$

are bounded.

8. Munuram owes Rs $10,000$ to a Bank. The bank charges an annual rate of 10%. Munuram makes payments continuously at a constant rate of Rs 100 per month (i.e., Rs 1200 per year). Set up the initial value problem for Munuram's loan and find the time T that it will take to pay off the debt.

9. Consider the metric d on $\mathbb{R}^2$ given by

$$d(x, y) = \begin{cases} \|x - y\| & \text{if } 0, x \text{ and } y \text{ are collinear,} \\ \|x\| + \|y\| & \text{otherwise,} \end{cases}$$

where $\| \cdot \|$ is the Euclidean norm. Prove that the metric space $(\mathbb{R}^2, d)$ has uncountably many pairwise disjoint open sets.

10. Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be continuous functions for $n \geq 1$. Suppose for each $x \in \mathbb{R}$, there exists $n(x) \in \mathbb{N}$ such that $f_{n(x)}(x) = x + n(x)$. Prove that there exists $a < b$ and $m \in \mathbb{N}$ such that $f_m(x) = x + m$ for all $x \in [a, b]$.