- $\mathbb{C}$ denotes the set of complex numbers.

- $\mathbb{R}$ denotes the set of real numbers.

- $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

1. For $x \in [0, \infty)$, define

$$f(x) = \sum_{n=0}^{\infty} \frac{e^{-nx}}{1 + n^2}.$$

   (a) Show that f is continuous on $[0, \infty)$.

   (b) Show that f is differentiable on $(0, \infty)$.

   (c) Does $f'(0+)$ exist finitely? Justify your answer.

2. A function $f : [a, b] \to \mathbb{R}$ is said to be a Lipschitz function if there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \text{for all } x, y \in [a, b].$$

   For two Lipschitz functions f and g on $[a, b]$, define

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| + \sup_{\substack{x,y \in [a,b] \\ x \neq y}} \frac{|(f-g)(x) - (f-g)(y)|}{|x - y|}.$$

   Show that the space of Lipschitz functions on $[a, b]$ with the metric d is a complete metric space.

3. Let $E \subset \mathbb{R}^n$ be a compact subset of positive Lebesgue measure.

   (a) If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then prove that the function $g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$g(y) = \int_E f(x + y)\, dx,$$

   is a continuous function.

   (b) If in (a), instead of continuity, we assume that f is Lebesgue integrable over compact sets, then does the conclusion that g is continuous still hold? Justify your answer.

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4. Let $f : \mathbb{R}^n \to [0, \infty)$ be a nonzero bounded Lebesgue integrable function satisfying the following:

(a) $f(x) = f(y)$ for all $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\|$.

(b) $f(x) \leq f(y)$ for all $x, y \in \mathbb{R}^n$ with $\|x\| \geq \|y\|$.

(c) For all $s \in [0, \infty)$, the set $B_s = \{x \in \mathbb{R}^n : f(x) > s\}$ is open in $\mathbb{R}^n$.

Prove that for every $t \in (0, f(\mathbf{0}))$,

$$B_t = B(\mathbf{0}, g(t))$$

for some nonnegative measurable function g on $(0, f(\mathbf{0}))$. Here $\mathbf{0}$ is the origin of $\mathbb{R}^n$ and $B(\mathbf{0}, g(t)) = \{x \in \mathbb{R}^n : \|x\| < g(t)\}$.

5. Let f be an entire function which is not a polynomial. If B is any bounded subset of $\mathbb{C}$, then prove that the set $f(\mathbb{C} \setminus B)$ is dense in $\mathbb{C}$.

6. (a) Define $f(z) = e^z - 1$ for all $z \in \mathbb{C}$. Prove that

$$\sup_{z \in \mathbb{C}} |f(z)| = f(1).$$

(b) Find the number of solutions of the equation $e^z = 2z + 1$, in the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

7. Let N be a positive integer. Find all real numbers a such that the differential equation

$$\frac{d^2 y}{dx^2} - 4a\frac{dy}{dx} + 3y = 0$$

has a nontrivial solution satisfying the conditions $y(0) = 0$ and $y(2N\pi) = 0$.

8. Let $(E, \|\cdot\|)$ be a normed linear space and ϕ a linear functional on E which is not continuous. Let $y_0 \in E$ be such that $\phi(y_0) = 1$. Define $S : E \to E$ by

$$S(x) = x - 2\phi(x)y_0.$$

Define a new norm on E by $\|x\|_\phi := \|Sx\|$.

(a) Show that $S^2 = I$, the identity operator on E.

(b) Show that there is a linear isomorphism $T : E \to E$ such that $\|T(x)\|_\phi = \|x\|$ for all $x \in E$.

(c) Show that $\|\cdot\|_\phi$ cannot be equivalent to $\|\cdot\|$.