PART I

Each question has 4 marks. Answer ALL questions.

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1. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

   (A) $\tan u$. (B) $\cot u$. (C) $\frac{1}{2}\tan u$. (D) $\frac{1}{2}\cot u$.

2. The value of the integral $\displaystyle\int_{-1}^{1} \frac{(x^2 + \lfloor x \rfloor)}{(x+1)} dx$, where $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u, is

   (A) 0. (B) 1/2. (C) $\log 2$. (D) $\log 2 - 2$.

3. Suppose A and B are two real $n \times n$ matrices with $n \geq 2$. Then which of the following two statements must be correct?

   (I) $rank(AB) = rank(BA)$.

   (II) $rank(A + B) \leq \min\{rank(A), rank(B)\}$.

   (A) None of (I) and (II). (B) Both (I) and (II).

   (C) Only (I). (D) Only (II).

4. The value of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+3)(2n+5)}$ is

   (A) $\log 2$. (B) $\frac{1}{6}$. (C) $\frac{1}{3}$. (D) $\frac{1}{2}$.

5. The value of $\displaystyle\sum_{n=1}^{2022} i^n$, where $i = \sqrt{-1}$, is

   (A) $-1 - i$. (B) $-1 + i$. (C) $1 + i$. (D) $1 - i$.

6. Consider a line $3x + 4y + 8 = 0$ which cuts a circle C at two points and forms a chord of length 10 cms. If the center of C is at $(3, 2)$, then the area of C is

(A) 100π. (B) 64π. (C) 50π. (D) 25π.

7. Consider the matrix

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 1 & 4 \\ 4 & 4 & 3 \end{bmatrix}.$$

If one of the eigenvalues of A is 3, the other two eigenvalues and corresponding eigenvectors of A are

(A) 8; $[2, -1, 1]^T$ and -2; $[-1, -1, 1]^T$.
(B) 8; $[1, 2, -2]^T$ and -2; $[2, 1, 2]^T$.
(C) 9; $[2, 1, 2]^T$ and -3; $[1, 2, -2]^T$.
(D) 9; $[1, 2, 1]^T$ and -3; $[2, -1, 0]^T$.

8. Let $\{f_n\}$ be the sequence of real numbers defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for any integer $n \geq 3$. If $a_n = f_{n+1}/f_n$, and if $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n$ equals

(A) $\frac{1+\sqrt{3}}{2}$. (B) $\frac{1+\sqrt{5}}{2}$. (C) $\frac{1-\sqrt{3}}{2}$. (D) $\frac{1-\sqrt{5}}{2}$.

9. Let f be a real valued function defined by

$$f(x) = \frac{e^{3x} - 3e^{x} + 2}{x^2}$$

for $x \neq 0$, and $f(x) = \alpha$ for $x = 0$. If f is differentiable at $x = 0$, then the value of α is

(A) 6. (B) 5. (C) 4. (D) 3.

10. The solution of $(2x - 2y + 7)dy + (y - x - 5)dx = 0$ is

(A) $2x - 2y + 3\log(x - y + 2) = x + C$.

(B) $x - y + 3\log(x - y + 2) = x + C$.

(C) $y - x + 3\log(y - x + 2) = x + C$.

(D) $2y - 2x + 3\log(y - x + 2) = x + C$.

PART II

Each question has 10 marks. Answer ALL questions.

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1. In a given fault plane, the principal stress magnitudes σ_1 and σ_3 are recorded as 50 MPa and 31 MPa at zero pore pressure. The angle between normal to fault plane and σ_1 is 70°. Determine the normal stress and shear stress. The rock has pre-existing fractures and angle of internal friction is 39°. The injection of water into the rock creates a pore pressure of 27 MPa. Show the position of new Mohr circle.

2. N_t, is the total number of particles in the sample at any given time t; and N_0 is the initial amount of active substance (at t=0). The half-life, $t_{\frac{1}{2}}$, is the time taken for the activity of a given amount of a radioactive substance to decay to half of its initial value. The decay constant λ is the reciprocal of the mean lifetime (s^{-1}). Consider the basic radioactive decay equation

$$dN = -\lambda N dt$$

where $N(t = 0) = N_0$ and $N(t) = N_t$.

Show that

(i) $N_t = N_0 \exp(-\lambda t)$, and

(ii) the half-life $t_{\frac{1}{2}} = \log \frac{2}{\lambda}$

Now, consider the case of a radioactive nuclide A that decays into nuclide B by some radioactive decay process $A \rightarrow B$ such that for any time t:

$$N_A + N_B = N_{total} = N_{A0}$$

N_{total} is constant throughout the decay process, and is equal to the initial number of A nuclides (N_{A0}) at $t = 0$. Also consider that at $t = 0$, $N_B = 0$. Show that the number of decayed nuclei B at time t is

(iii) $N_B = N_{A0}(1 - e^{-\lambda t})$

3. The results of the modal analysis for 1000 points from a sandstone sample are given in the table.

Mineral	Count
Quartz	300
Plagioclase	250
Orthoclase	150
Rock Fragments	200
Clay	100

Name the sandstone based on Pettijohn, Potter and Siever's scheme of classification. Plot qualitatively the position on a CFL diagram.

4. A large area is intruded by three basaltic sills with thickness of 30, 40, and 50 m. The basaltic sill density is 2.8 Mg/m^3 and the asthenosphere density is 3.2 Mg/m^3. What is the change in the height of the surface after isostatic equilibrium has been restored?

5. A paleontologist found fossils of trilobites without any spine. He prepared a table of the lengths from the tip of the cephalon to the end of the pygidium (X) and the maximum breadth of the cephalon at the cheeks (Y). Below is the table, the person prepared.

X (in cm)	Y (in cm)
3	1.4
4	1.5
5	2.2
6	2.4
8	3.1
9	3.2
10	3.2
11	3.9
12	4.1
14	4.7
15	4.5
16	5.2
17	5.0

Now, find out the equation of the best fit line for the lengths of the trilobites as to their breadths. (Note that $\Sigma XY = 514.8$; $\Sigma X \Sigma Y = 5772$; $\Sigma X^2 = 1562$).

6. A 2D circular marker has been deformed to become an ellipse having 32 cm and 8 cm as major and minor axes. Find the radius of the original undeformed marker, assuming volume constant homogeneous deformation.
