Notation: Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and ln be the logarithm to the base e.

1. If $p \in \mathbb{R}$ and $f(x) = (p^2 + p - 6)\cos 2x + (p-2)x + \cos 1$ is the real valued function, then the set of values of p for which $f'(x) = 0$ has no solution is

   (A) $(-\infty, -\frac{5}{2})$. (B) $(-\frac{7}{2}, \infty)$.
   (C) $(-\infty, -2]$. (D) $(-\frac{7}{2}, -\frac{5}{2})$.

2. Suppose matrices $A_{n\times m}$ and $B_{m\times r}$ are such that $rank(B) = r$. Then which of the following two statements must be correct?

   (I) $rank(A) = rank(AA^T)$.

   (II) $rank(A) = rank(AB)$.

   (A) Both (I) and (II). (B) Only (I).
   (C) Only (II). (D) None of (I) and (II).

3. An integer is chosen at random from the set $\{1, 2, \ldots, 100\}$. The probability that the chosen integer is divisible by 2 or 3 or 5 is

   (A) 0.80. (B) 0.74. (C) 0.71. (D) 0.64.

4. The point on the graph $y = 2\sqrt{x}$ which is nearest to the point $(3, 0)$ is

   (A) $(0, 0)$. (B) $(2, 2\sqrt{2})$. (C) $(1, 2)$. (D) $(3, 2\sqrt{3})$.

5. The best approximation for the value of $\displaystyle\int_0^{\frac{1}{2}} \cos(x^2)dx$, accurate to within 0.0001, is

   (A) $\frac{1}{2} + \frac{1}{320}$. (B) $\frac{1}{2} + \frac{1}{640}$. (C) $\frac{1}{2} - \frac{1}{640}$. (D) $\frac{1}{2} - \frac{1}{320}$.

6. The area of the region enclosed by the curves $xy = 2$, $2y = x$ and $y = 2$ is

(A) $2\ln 2 - 1$.   (B) $2\ln 2 + 1$.

(C) $3 - 2\ln 2$.   (D) $3 + 2\ln 2$.

7. Let f be a function on $\mathbb{R}$ such that $|f'(x)| \leq \sqrt{x}$ for all $x \in \mathbb{R}$. Then $\lim_{x\to\infty} \dfrac{f(x+2) - f(x)}{x}$ equals

(A) -2.   (B) 0.   (C) $\sqrt{2}$.   (D) 2.

8. The mean of the random variable whose probability density function is given by

$$f(x) = \begin{cases} \frac{2}{3}(x - \lfloor x \rfloor) & \text{if } -1 \leq x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lfloor u \rfloor$ is the greatest integer less than or equal to u, is

(A) $\frac{1}{2}$.   (B) $\frac{2}{3}$.   (C) 1.   (D) $\frac{3}{2}$.

9. The number of bijective functions from $\{1, 2, 3, 4, 5\}$ to itself such that $f(4) \neq 4$ and $f(5) \neq 5$ is

(A) 72.   (B) 78.   (C) 102.   (D) 114.

10. In $\mathbb{R}$, the equation $2\sin x + 3\cos x - x = 0$ has

(A) at least three solutions.   (B) exactly two solutions.

(C) exactly one solution.   (D) no solution.

11. Let α be a real number such that the difference between the real zeros of the polynomial $\mathcal{P}(x) = x^2 - \alpha x + \alpha - 9$ is equal to 6. Then which one of the following is NOT a possible zero of $\mathcal{P}$?

(A) -5. (B) -3. (C) 3. (D) 5.

12. In a factory, each batch of items is examined by two robots. When a defective batch comes, the first robot detects it with probability 0.4. Among the batches that get past the first robot, the second robot detects a defective batch with probability 0.5. If a defective batch is finally detected, the probability that it was detected by the second robot is

(A) $\frac{3}{10}$. (B) $\frac{4}{10}$. (C) $\frac{3}{7}$. (D) $\frac{4}{7}$.

13. Let $\mathcal{S}$ be the set of positive continuous functions $f: [0,1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)dx = 1, \int_0^1 xf(x)dx = \alpha \text{ and } \int_0^1 x^2 f(x)dx = \alpha^2,$$

where α is a nonnegative real number. Then $\mathcal{S}$ is

(A) an uncountable set. (B) a countably infinite set.
(C) a non-empty finite set. (D) an empty set.

14. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Which of the following implies that $f(x) = 0$ for all $x \in \mathbb{R}$?

(A) $\int_a^{a+1} f(x)dx = 0$ for all $a \in \mathbb{R}$.
(B) $\int_a^{a+\frac{1}{n}} f(x)dx = 0$ for all $a \in \mathbb{R}$ and for all $n \in \mathbb{N}$.
(C) $\int_{-|a|}^{|a|+\frac{1}{n}} f(x)dx = 0$ for all $a \in \mathbb{R}$ and for all $n \in \mathbb{N}$.
(D) $\int_{-a}^{a} f(x)dx = 0$ for all $a > 0$.

15. Suppose that f is a continuous function on the positive real line, and that for all $x, y > 0$, $\int_x^{xy} f(t)dt$ does not depend on x. If $f(2) = 2$, then for $x > 0$, the value of $\int_1^x f(t)dt$ is

(A) $4\ln x$. (B) $2\ln x$. (C) $\ln x$. (D) $\frac{1}{2}\ln x$.

16. If a, b and c are positive real numbers such that $abc = 9$, then the minimum value of $(a+bc)(b+ca)(c+ab)$ is

(A) 27. (B) 162. (C) 207. (D) 216.

17. If $A = \{x \in \mathbb{R} : \frac{|x-3|}{|x-2|} < 3\}$, then A is equal to

(A) $(-\infty, \frac{3}{2}) \cup (2, \infty)$. (B) $(-\infty, 2) \cup (\frac{9}{4}, \infty)$.
(C) $(-\infty, \frac{3}{2}) \cup (\frac{9}{4}, \infty)$. (D) $(\frac{9}{4}, \infty)$.

18. The value of $\lim_{x\to\infty} |\sin(\pi\sqrt{x^2+x+1})|$ is

(A) 1. (B) $\frac{1}{\sqrt{2}}$. (C) $\frac{1}{2}$. (D) 0.

19. If α and β are the roots of the equation $x^2+x+1=0$, then the value of $\alpha^{2022} + \beta^{2022}$ is

(A) -2. (B) -1. (C) 1. (D) 2.

20. Let f be a real valued continuous function such that

$$f(x) = x^2 + \int_{-1}^{1} (x+y)f(y)dy.$$

The value of the integral $\int_{-1}^{1} f(t)dt$ is

(A) $\frac{2}{3}$. (B) 0. (C) -2. (D) $-\frac{10}{3}$.

21. The value of $\int_1^3 \left|\frac{x^2-3x+2}{x^2+2x-3}\right| dx$ is

(A) $5\ln(\frac{25}{24})-1$. (B) $2-5\ln(\frac{6}{4})$.

(C) $5\ln(\frac{6}{4})-2$. (D) $5\ln(\frac{25}{24})$.

22. The possible values of λ and μ so that the system of equations $p+2q+r=7$, $p+q+\lambda r=\mu$ and $p+3q-5r=5$ has no solution are, respectively,

(A) 6 and 9. (B) 7 and 6.

(C) 6 and 7. (D) 9 and 7.

23. The polynomial $x^5-10x+20$ has

(A) only negative real roots.

(B) only positive real roots.

(C) both positive and negative real roots.

(D) at least two complex roots.

24. The series $\displaystyle\sum_{n=2}^{\infty}\frac{(x+3)^n}{(-2)^n(n\ln n+1)}$ converges for all the values of x in the interval

(A) $(-2,-1]$. (B) $(-5,1]$.

(C) $(-5,-1]$. (D) $(-2,1]$.

25. The value of $\lim\limits_{n\to\infty}(2^{n+8\sqrt{n}}+3^{n+4\sqrt{n}}+4^{n+2\sqrt{n}}+5^{n+\sqrt{n}})^{\frac{1}{n}}$ is

(A) 2^8. (B) 3^4. (C) 4^2. (D) 5.

26. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -2x & -4 \\ 0 & 0 & -5 \end{bmatrix}$ are matrices such that the trace of AB equals zero, then the value of x is

(A) $\frac{13}{4}$. (B) $\frac{5}{4}$. (C) $-\frac{5}{4}$. (D) $-\frac{13}{4}$.

27. Suppose p, q and r are real numbers such that the matrix

$$A = \frac{1}{7}\begin{bmatrix} 6 & -3 & p \\ q & -2 & 6 \\ 2 & r & 3 \end{bmatrix}$$

is orthogonal. If $i = \sqrt{-1}$, then the value of the determinant of A is

(A) $49\sum_{n=1}^{23} i^n$. (B) $49\sum_{n=1}^{24} i^n$. (C) $49\prod_{n=1}^{23} i^n$. (D) $49\prod_{n=1}^{22} i^n$.

28. If (a_n) is a sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

(A) $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges if $a_n > 0$ for all $n \in \mathbb{N}$.

(B) $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent.

(C) $\sum_{n=1}^{\infty} a_n^2$ is convergent.

(D) $\sum_{n=1}^{\infty} a_n b_n$ converges for any sequence (b_n) converging to 1.

29. Suppose a circle is inscribed in the square specified by the lines $x = 3$, $x = 6$, $y = 5$ and $y = 8$, then the circumference of the circle is

(A) π. (B) 2π. (C) 3π. (D) 4π.

30. Let $S = \{1, 2, \ldots, 100\}$. The number of subsets of S having exactly three distinct elements which add up to an odd integer is

(A) $\binom{100}{3}$.

(B) $50 \times \binom{50}{2}$.

(C) $100 \times \binom{50}{2}$.

(D) $\frac{1}{2} \times \binom{100}{3}$.