Notations: Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and ln be the logarithm to the base e.

1. The number of continuous functions f on $\mathbb{R}$ which satisfy the equation $(f(x))^2 = x^2$ for all $x \in \mathbb{R}$ is

   (A) 2. (B) 4. (C) 8. (D) uncountable.

2. Let $f(x) = x^k \cos \frac{1}{x}$, if $x > 0$, and $f(x) = 0$, otherwise. Then the set of values of k for which f is differentiable in $\mathbb{R}$ is

   (A) $(0,\ \infty)$. (B) $(1,\ \infty)$.
   (C) $(2,\ \infty)$. (D) $\mathbb{R} \setminus \{0\}$.

3. Suppose f is continuous on $[1,\ 6]$ and the only solutions of the equation $f(x) = 7$ are $x = 1$ and $x = 5$. If $f(3) = 9$, then the possible value of the pair $(f(2), f(4))$ is

   (A) $(5.2,\ 9.2)$. (B) $(5.2,\ 8.8)$.
   (C) $(7.2,\ 9.2)$. (D) $(7.2,\ 8.8)$.

4. If n terms $a_1, a_2, \ldots, a_n$ are in arithmetic progression with increment r, then the difference between the mean of their squares and the square of their mean is

   (A) $\frac{r^2}{12}(n^2 - 1)$. (B) $\frac{r^2}{12}(n^2 + 1)$.
   (C) $\frac{r^2}{2}(n^2 + 1)$. (D) $\frac{r^2}{2}(n^2 - 1)$.

5. Let $A = \{x \in \mathbb{R} : \frac{x+2}{x-1} < 4\}$. Then A is equal to

   (A) $\mathbb{R} \setminus \{1, 2\}$. (B) $(-\infty,\ \frac{2}{3}) \cup (1,\ \infty)$.
   (C) $(-\infty,\ \frac{2}{3}) \cup (2,\ \infty)$. (D) $(-\infty,\ 1) \cup (2,\ \infty)$.

6. Let $S = \{1, 2, \ldots, n\}$. The number of sets of the form $\{A, B\}$ where A and B are disjoint subsets of S, such that both A and B cannot be empty, is

(A) 3^n. (B) $\frac{3^n-1}{2}$. (C) $\frac{3^n+1}{2}$. (D) $3^n - 1$.

7. The number of ways in which 10 distinct balls can be placed in 10 distinct boxes (each box can contain maximum 10 balls) so that exactly one box remains empty is

(A) $45 \times 9!$. (B) $90 \times 9!$. (C) $45 \times 10!$. (D) $90 \times 10!$.

8. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$, $Q = PAP^T$, then $P^T Q^{2021} P$ is

(A) $\begin{bmatrix} -1 & -2021 \\ 0 & -1 \end{bmatrix}$. (B) $\begin{bmatrix} 1 & 2021 \\ 0 & 1 \end{bmatrix}$.

(C) $\begin{bmatrix} -1 & 2021 \\ 0 & -1 \end{bmatrix}$. (D) $\begin{bmatrix} 1 & -2021 \\ 0 & 1 \end{bmatrix}$.

9. Let $T_1 = (0, 1)$, $T_2 = (0, \frac{1}{2})$, $T_3 = (0, \frac{1}{3})$, $\ldots$ be a sequence of open intervals on the real line. Then $\bigcap_{k \in \mathbb{N}} T_k$ is

(A) $\emptyset$. (B) $\{0\}$.

(C) $(0, \frac{1}{n})$ for some $n \in \mathbb{N}$. (D) $\{\frac{1}{n}\}$ for some $n \in \mathbb{N}$.

10. The sum of the roots of the equation $\lfloor 3x + 1 \rfloor = 2x - \frac{1}{2}$, where $\lfloor u \rfloor$ is the greatest integer less than or equal to u, is

(A) -3. (B) -2. (C) $\frac{1}{2}$. (D) $-\frac{3}{4}$.

11. Let $p(x)$ be a polynomial function with real coefficients. It is given that for all real numbers, x, we have $|p(x)| \leq 2021$. If $p(2020) = \frac{1}{2020}$, then the value of $p(2021)$ is

(A) 2021. (B) $\frac{1}{2021}$.

(C) $\frac{1}{2020}$. (D) not possible to determine.

12. Let $A = \{(a, b, c, d, e) : a+b+c = 5,\ d+e \leq 4,\ a, b, c, d, e \in \mathbb{N} \cup \{0\}\}$. Then the number of elements of A is

(A) 210. (B) 60. (C) 315. (D) 441.

13. The value of $\int_1^2 \lfloor x^2 - 1 \rfloor dx$, where $\lfloor u \rfloor$ is the greatest integer less than or equal to u, is

(A) $4 + \sqrt{3} - \sqrt{2}$. (B) $4 - \sqrt{3} - \sqrt{2}$.

(C) $4 - \sqrt{3} + \sqrt{2}$. (D) $4 + \sqrt{3} + \sqrt{2}$.

14. Call a natural number *wonderful* if it is not divisible by the cube of any prime number. The maximum possible number of consecutive wonderful numbers is

(A) 5. (B) 6. (C) 7. (D) 8.

15. For a group of 10 people, the expected number of days of the year which are birthdays of exactly 2 people (assuming that none of the persons in the group were born in a leap year and that each date is equally likely to be a birthday for each person) is

(A) $\frac{45\,(364)^8}{(365)^{10}}$. (B) $45(\frac{364}{365})^8$. (C) $\frac{45\,(364)^8}{(365)^9}$. (D) $45(\frac{364}{365})^9$.

16. Let $p(x)$ be a real polynomial of degree 2021. If $p(n)p(n+1) < 0$ for $n \in \{1, 2, ..., 2021\}$, then which of the following *must* be true?

(A) $p(0)p(2023) > 0$. (B) $p(0)p(2023) < 0$.

(C) $p(0)p(2023) > 1$. (D) $p(0)p(2023) < -1$.

17. The value of the series $\sum_{n=0}^{\infty}[(2n+1)(2n+2)]^{-1}$ is

(A) $\ln 2 - 1$. (B) $2\ln 2 - 1$. (C) $2\ln 2$. (D) $\ln 2$.

18. Three numbers are selected at random without replacement from $\{1, \ldots, n\}$ for $n \in \mathbb{N}$. The probability that the third number falls between the other two numbers given that the first number is smaller than the second number is

(A) $\frac{1}{2}$. (B) $\frac{1}{6}$. (C) $\frac{1}{4}$. (D) $\frac{1}{3}$.

19. Let $y = f(x)$ be a curve. Then, to obtain the graph of $y = f(\frac{1+x}{2})$, it is enough to change the graph of $y = f(x)$ by horizontally shifting it

(A) to the right by $\frac{1}{2}$ and stretching by a factor of 2.

(B) to the left by $\frac{1}{2}$ and stretching by a factor of $\frac{1}{2}$.

(C) to the right by $\frac{1}{2}$ and stretching by a factor of $\frac{1}{2}$.

(D) to the left by $\frac{1}{2}$ and stretching by a factor of 2.

20. If the equation $x^2 + 2(3-a)y - by^2 - 2ax = 3ab$ represents a circle, then the set of values of a *must* be

(A) $(\frac{3}{2},\ 3)$. (B) $(-\infty,\ -\frac{3}{2}) \bigcup\ (3,\ \infty)$.

(C) $(-\infty,\ \frac{3}{2}) \bigcup\ (3,\ \infty)$. (D) $(-\infty,\ -3) \bigcup\ (\frac{3}{2},\ \infty)$.

21. The values of $n \in \mathbb{N}$ for which the polynomial $x^n + (x-1)^n + (2x-1)^n - (3^n + 2^n + 1)$ is divisible by $x^2 - x - 2$ are

(A) only $\{2, 4\}$.

(B) only $\{2, 4, 6\}$.

(C) only $\{2, 4, 8\}$.

(D) all even numbers.

22. If $a > 0$, then $\lim_{x \to a} \frac{x^a - a^x}{x^x - a^a}$ equals

(A) $\frac{1+\ln a}{1-\ln a}$.

(B) $\frac{1-\ln a}{1+\ln a}$.

(C) $\frac{1+a\ln a}{1-a\ln a}$.

(D) $\frac{1-a\ln a}{1+a\ln a}$.

23. Let $I = [a, b]$ for $a < b$ and let $g : I \to \mathbb{R}$ be continuous on I. Then which of the following statements are correct?

(i) $\{g(x) : a < x \leq b\}$ is an interval not necessarily closed and bounded.

(ii) g may not have an absolute minimum or maximum on I.

(A) only (i).

(B) only (ii).

(C) both (i) and (ii).

(D) none of the (i) and (ii).

24. If $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$, then which of the following statements are correct?

(i) f is decreasing on $(-\infty, \ -1)$.

(ii) f is increasing on $(-1, \ 0)$.

(iii) f is decreasing on $(0, \ 2)$.

(iv) f is increasing on $(2, \ \infty)$.

(A) only (i) and (iv).

(B) only (i) and (ii).

(C) only (i), (iii) and (iv).

(D) all the four statements.

25. The value of $\lim_{n \to \infty} (n!)^{n^{-2}}$ is

(A) 0.

(B) 1.

(C) e.

(D) ∞.

26. The number of ways to colour an n-sided regular prism ($n \geq 5$) using $n+2$ colours such that each face gets a unique colour is

(A) $\frac{1}{2}(n+2)(n+1)(n-1)!$. (B) $(n+2)(n+1)(n-1)!$.

(C) $\frac{1}{2}(n+2)(n+1)n!$. (D) $(n+2)(n+1)n!$.

27. Suppose that $A_1, A_2, \ldots, A_n$ are mutually disjoint events. If B is an event such that $P(B|A_i) = 1/3$ and $P(A_i) > 0$ for all $i = 1, \ldots, n$, then $P(B|A)$ where $A = \cup_{i=1}^{n} A_i$ is

(A) $\frac{1}{2n}$. (B) $\frac{1}{3}$. (C) $\frac{1}{3n}$. (D) $\frac{1}{2}$.

28. If $\omega^3 = 1$ but $\omega \neq 1$ and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^{2021} - 1 & \omega^{2024} \\ 1 & \omega^{2024} & \omega^{2020} \end{vmatrix} = 3k,$$

then k is equal to

(A) $\pm i\sqrt{3}$. (B) $\pm\frac{i\sqrt{3}}{2}$. (C) $1 \pm i\sqrt{3}$. (D) $\frac{1 \pm i\sqrt{3}}{2}$.

29. Suppose $f(x)$ is a twice differentiable function on $[0, \pi]$. If $f(\pi) = 10$ and $\int_0^{\pi} \{f(x) + f''(x)\} \sin x \, dx = 21$, then the value of $f(0)$ is

(A) 0. (B) 10. (C) 11. (D) 21.

30. The value of $\int_0^{\pi/3} \{g(x) + g(\frac{\pi}{3} - x)\}^{-1} g(x) \, dx$, where g is a function such that the integral exists, is

(A) $\pi/6$. (B) $\pi/3$. (C) $g(\pi/6)$. (D) $g(\pi/3)$.