1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $|f'(x)| < \frac{1}{2}$ for all $x \in \mathbb{R}$. Show that for some $x_0 \in \mathbb{R}$, $f(x_0) = x_0$. [10 points]

2. If the interior angles of a triangle ABC satisfy the equality,
$$\sin^2 A + \sin^2 B + \sin^2 C = 2(\cos^2 A + \cos^2 B + \cos^2 C),$$
prove that the triangle must have a right angle. [10 points]

3. Suppose $f : [0,1] \to \mathbb{R}$ is differentiable with $f(0) = 0$. If $|f'(x)| \leq f(x)$ for all $x \in [0,1]$, then show that $f(x) = 0$ for all x. [10 points]

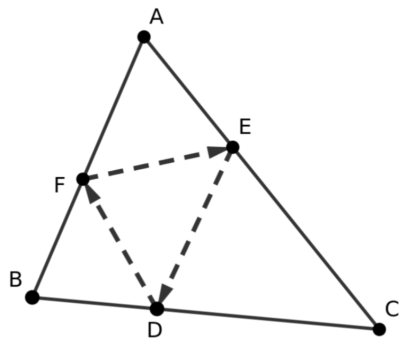
4. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane. Let $f : S^1 \to S^1$ be the map given by $f(z) = z^2$. We define $f^{(1)} := f$ and $f^{(k+1)} := f \circ f^{(k)}$ for $k \geq 1$. The smallest positive integer n such that $f^{(n)}(z) = z$ is called the *period* of z. Determine the total number of points in S^1 of period 2025. (Hint: $2025 = 3^4 \times 5^2$) [10 points]

5. Let a, b, c be nonzero real numbers such that $a + b + c \neq 0$. Assume that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}.$$
Show that for any odd integer k,
$$\frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k} = \frac{1}{a^k + b^k + c^k}.$$
[10 points]

6. Let $\mathbb{N}$ denote the set of natural numbers, and let (a_i, b_i), $1 \leq i \leq 9$, be nine distinct tuples in $\mathbb{N} \times \mathbb{N}$. Show that there are three distinct elements in the set $\{2^{a_i} 3^{b_i} : 1 \leq i \leq 9\}$ whose product is a perfect cube.

[10 points]

7. Consider a ball that moves inside an acute-angled triangle along a straight line, until it hits the boundary, which is when it changes direction according to the mirror law, just like a ray of light (angle of incidence = angle of reflection). Prove that there exists a triangular periodic path for the ball, as pictured below.



[10 points]

8. Let $n \geq 2$ and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers such that $\sum_{i=1}^{n} a_i = \Pi_{i=1}^{n} a_i$. Prove that $\sum_{i=1}^{n} a_i \leq 2n$ and determine when equality holds. [10 points]