

**Q1.** Find, with proof, all possible values of  $t$  such that

$$\lim_{n \rightarrow \infty} \left\{ \frac{1 + 2^{1/3} + 3^{1/3} + \dots + n^{1/3}}{n^t} \right\} = c$$

for some real number  $c > 0$ . Also find the corresponding values of  $c$ .

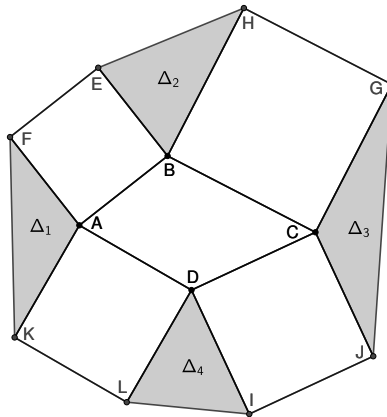
**Q2.** Suppose  $n \geq 2$ . Consider the polynomial

$$Q_n(x) = 1 - x^n - (1 - x)^n.$$

Show that the equation  $Q_n(x) = 0$  has only two real roots, namely 0 and 1.

**Q3.** Let  $ABCD$  be a quadrilateral with all internal angles  $< \pi$ . Squares are drawn on each side as shown in the picture below. Let  $\Delta_1, \Delta_2, \Delta_3$  and  $\Delta_4$  denote the areas of the shaded triangles shown. Prove that

$$\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = 0.$$



**Q4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function which is differentiable at 0. Define another function  $g : \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$g(x) = \begin{cases} f(x) \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose that  $g$  is also differentiable at 0. Prove that

$$g'(0) = f'(0) = f(0) = g(0) = 0.$$

**Q5.** Let  $P(x)$  be a polynomial with real coefficients. Let  $\alpha_1, \dots, \alpha_k$  be the distinct real roots of  $P(x) = 0$ . If  $P'$  is the derivative of  $P$ , show that for each  $i = 1, 2, \dots, k$ ,

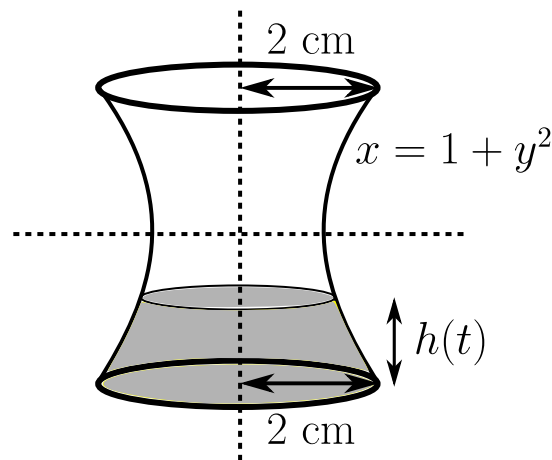
$$\lim_{x \rightarrow \alpha_i} \frac{(x - \alpha_i)P'(x)}{P(x)} = r_i,$$

for some positive integer  $r_i$ .

**Q6.** Let  $x_1, \dots, x_{2024}$  be non-negative real numbers with  $\sum_{i=1}^{2024} x_i = 1$ . Find, with proof, the minimum and maximum possible values of the expression

$$\sum_{i=1}^{1012} x_i + \sum_{i=1013}^{2024} x_i^2.$$

**Q7.** Consider a container of the shape obtained by revolving a segment of the parabola  $x = 1 + y^2$  around the  $y$ -axis as shown below. The container is initially empty. Water is poured at a constant rate of  $1 \text{ cm}^3/\text{s}$  into the container. Let  $h(t)$  be the height of water inside the container at time  $t$ . Find the time  $t$  when the rate of change of  $h(t)$  is maximum.



**Q8.** In a sports tournament involving  $N$  teams, each team plays every other team exactly once. At the end of every match, the winning team gets 1 point and the losing team gets 0 points. At the end of the tournament, the total points received by the individual teams are arranged in decreasing order as follows:

$$x_1 \geq x_2 \geq \cdots \geq x_N.$$

Prove that for any  $1 \leq k \leq N$ ,

$$\frac{N-k}{2} \leq x_k \leq N - \frac{k+1}{2}.$$