Q 1.

Determine all integers $n > 1$ such that every power of n has an odd number of digits.

Q 2.

Let $a_0 = \frac{1}{2}$ and a_n be defined inductively by

$$a_n = \sqrt{\frac{1 + a_{n-1}}{2}}\,, n \geq 1\,.$$

(a) Show that for $n = 0, 1, 2, \ldots$,

$$a_n = \cos\theta_n \text{ for some } 0 < \theta_n < \frac{\pi}{2}\,,$$

and determine θ_n.

(b) Using (a) or otherwise, calculate

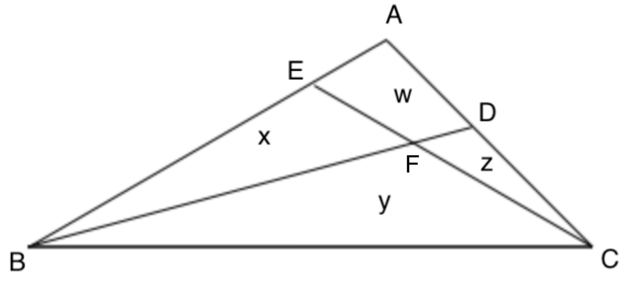
$$\lim_{n\to\infty} 4^n \left(1 - a_n\right)\,.$$

Q 3.

In a triangle ABC, consider points D and E on AC and AB, respectively, and assume that they do not coincide with any of the vertices A, B, C. If the segments BD and CE intersect at F, consider the areas w, x, y, z of the quadrilateral $AEFD$ and the triangles BEF, BFC, CDF, respectively.

(a) Prove that $y^2 > xz$.

(b) Determine w in terms of x, y, z.



Q 4.

Let $n_1, n_2, \cdots, n_{51}$ be distinct natural numbers each of which has exactly 2023 positive integer factors. For instance, 2^{2022} has exactly 2023 positive integer factors $1, 2, 2^2, \cdots, 2^{2021}, 2^{2022}$. Assume that no prime larger than 11 divides any of the n_i's. Show that there must be some perfect cube among the n_i's. You may use the fact that $2023 = 7 \times 17 \times 17$.

Q 5.

There is a rectangular plot of size $1 \times n$. This has to be covered by three types of tiles - red, blue and black. The red tiles are of size 1×1, the blue tiles are of size 1×1 and the black tiles are of size 1×2. Let t_n denote the number of ways this can be done. For example, clearly $t_1 = 2$ because we can have either a red or a blue tile. Also, $t_2 = 5$ since we could have tiled the plot as: two red tiles, two blue tiles, a red tile on the left and a blue tile on the right, a blue tile on the left and a red tile on the right, or a single black tile.

(a) Prove that $t_{2n+1} = t_n(t_{n-1} + t_{n+1})$ for all $n > 1$.

(b) Prove that $t_n = \sum_{d \geq 0} \binom{n-d}{d} 2^{n-2d}$ for all $n > 0$.

Here,

$$\binom{m}{r} = \begin{cases} \frac{m!}{r!(m-r)!}\,, & \text{if } 0 \leq r \leq m\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

for integers m, r.

Q 6.

Let $\{u_n\}_{n \geq 1}$ be a sequence of real numbers defined as $u_1 = 1$ and

$$u_{n+1} = u_n + \frac{1}{u_n} \text{ for all } n \geq 1\,.$$

Prove that $u_n \leq \frac{3\sqrt{n}}{2}$ for all n.

Q 7.

(a) Let $n \geq 1$ be an integer. Prove that $X^n + Y^n + Z^n$ can be written as a polynomial with integer coefficients in the variables $\alpha = X + Y + Z$, $\beta = XY + YZ + ZX$ and $\gamma = XYZ$.

(b) Let $G_n = x^n \sin(nA) + y^n \sin(nB) + z^n \sin(nC)$, where x, y, z, A, B, C are real numbers such that $A + B + C$ is an integral multiple of π. Using (a) or otherwise, show that if $G_1 = G_2 = 0$, then $G_n = 0$ for all positive integers n.

Q 8.

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which is differentiable on $(0, 1)$. Prove that either f is a linear function $f(x) = ax + b$ or there exists $t \in (0, 1)$ such that $|f(1) - f(0)| < |f'(t)|$.